S.-T. Yau College Student Mathematics Contests 2021

Oral Exams in Geometry and Topology

Individual (4 problems)

1. Consider the manifold

$$M = \left\{ \left((x_1, \cdots, x_n), [y_1 : \cdots : y_n] \right) \in \mathbb{R}^n \times \mathbb{RP}^{n-1} : x_i y_j = x_j y_i \ \forall i, j \right\},$$

and the projection map $\pi : M \to \mathbb{R}^n$ onto its $\mathbb{R}^n$-factor. Determine whether or not π is a submersion.

2. The suspension SX of a topological space X is defined as the quotient space of $X \times [0,1]$ modulo the equivalence relation generated by

$$(x_1, 0) \sim (x_2, 0) \text{ and } (x_1, 1) \sim (x_2, 1) \text{ for all } x_1, x_2 \in X.$$

Show that for all n there are isomorphisms $\widetilde{H}_n(SX) = \widetilde{H}_{n-1}(X)$.

3. Let M be a compact orientable Riemannian manifold with nonnegative Ricci curvature. Then prove the following:

(a) The first Betti number $b_1(M) \leq \dim M$.
(b) The above equality holds if and only if M is isometric to a flat torus.
(c) If we further assume M has positive Ricci curvature, then $b_1(M) = 0$.

4. Let M be a Riemannian manifold, let $p \in M$, and let Π be a plane in T_pM (i.e., a 2-dimensional linear subspace of T_pM). Let $D_r \subset T_pM$ be the open disc of radius r in the plane Π, centered at 0. For r sufficiently small, we know that $\exp_p(D_r)$ is an embedded 2-dimensional submanifold of M; call its area A_r. Prove that the sectional curvature

$$K(\Pi) = \lim_{r \to 0+} 12 \frac{\pi r^2 - A_r}{\pi r^4}.$$

If you could not give a general proof, maybe try when M is surface, i.e., $\dim M = 2$.